A **first-order first-degree differential equation** is a differential equation that is both a first-order differential equation and a first-degree differential equation.

A DE of the type $M + N\frac{dy}{dx} = 0$

where M and N are functions of x and y or constants is called a DE of 1^{st} order and 1^{st} degree.

Variables Separated From (V.S.F.)

Simply put, a differential equation is said to be **separable** if the variables can be separated. We can solve this by separating the equation into two parts. We move all of the equation involving the y variable to one side and all of the equation involving the x variable to the other side, then we can integrate both sides.

That is, a separable equation is one that can be written in the form

$$f(y)\, dy = g(x)\, dx$$

Once this is done, all that is needed to solve the equation is to integrate both sides.

Example 1: Solve the equation $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$

Solution:

Separating the variables the equation becomes

$$\frac{dy}{1+y^2} = \frac{dx}{1+x^2}$$

Then integrate we get,

$$\int \frac{dy}{1+y^2} = \int \frac{dx}{1+x^2}$$

$$\Rightarrow \tan^{-1} y = \tan^{-1} x + c$$

$$\Rightarrow \tan^{-1} y - \tan^{-1} x = c$$

$$\Rightarrow \tan^{-1} \frac{y-x}{1+xy} = \tan^{-1} c$$

[Taking, $c = \tan^{-1} c$]

$$\Rightarrow \frac{y-x}{1+xy} = c$$

$$\Rightarrow y - x = c(1 + xy)$$

Example 2:

Solve DE $\mathbf{2ydy = (x^2 + 1)dx}$

Since this equation is already expressed in "separated" form, just integrate:

$$2ydy = (x^2 + 1)dx$$

$$\Rightarrow \int 2ydy = \int (x^2 + 1)dx$$

$$\Rightarrow y^2 = \frac{x^3}{3} + x + c$$

Solve: $\mathbf{(e^y + 1)\cos x\, dx + e^y \sin x\, dy = 0}$

Separating the variables the equation becomes

$$(e^y + 1)\cos x\, dx = -e^y \sin x\, dy$$

$$\Rightarrow \frac{\cos x}{\sin x} dx = -\frac{e^y}{1+e^y} dy$$

$$\Rightarrow \cot x\, dx = -\frac{e^y}{1+e^y} dy$$

Then integrate we get,

$$\int \cot x\, dx = \int -\frac{e^y}{1+e^y} dy$$

$$\Rightarrow \ln \sin x = \int -\frac{e^y}{1+e^y} dy$$

Put, $1 + e^y = z \Rightarrow e^y dy = dz$ then

$$\ln \sin x = \int -\frac{dz}{z}$$

$$\Rightarrow \ln \sin x = -\ln z + c$$

$$\Rightarrow \ln \sin x = -\ln z + \ln c$$

$$\Rightarrow \ln \sin x = -\ln(1 + e^y) + \ln c$$

[put $\ln c = c$]

$$\Rightarrow \ln \sin x + \ln(1 + e^y) = \ln c$$

$$\Rightarrow \ln\{\sin x\, (1 + e^y)\} = \ln c$$

$$\Rightarrow \sin x\, (1 + e^y) = c$$

Exercise :

Solve the following equations

(i) $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$	(vii) $\frac{dy}{dx} = x\, e^{2y}$
(ii) $\frac{dy}{dx} = e^{2x-3y} + x^2 e^{-3y}$	(viii) $\frac{dy}{dx} = 2x(1-y)^2$
(iii) $e^{x-y} dx + e^{y-x} dy = 0$	(ix) $(e^y + 1)\cos x\, dx + e^y \sin x\, dy = 0$
(iv) $\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0$	(x) $(y+1)\cos x\, dx - dy = 0$
(v) $\left(y - \frac{dy}{dx}\right) x = y$	(xi) $(y-1)\sin x\, dx - dy = 0$
(vi) $y - x\frac{dy}{dx} = a(y^2 + \frac{dy}{dx})$	(xii) $(3 + 2\sin x + \cos x)dy = (1 + 2\sin y + \cos y)dx$
	(xiii) $(y - yx)dx + (x + xy)dy = 0$
	(xiv) $3e^x \tan y\, dx + (1 - e^{-x})\sec^2 y\, dy = 0$